Efficient Analysis of Waveguide Components Using a Hybrid PEE-FDFD Algorithm

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***Abstract*—The partial eigenfunction expansion (PEE) method combined with the classical finite difference frequency domain (FDFD) algorithm is proposed to accelerate frequency domain analysis of waveguide components. Examples are shown validating the method both for eigenvalue and deterministic problems.**

***Index Terms*—Finite difference frequency domain (FDFD), modal expansion, waveguides.**

TECHNIQUES based on Yee's scheme of finite difference discretization of Maxwell's equations are among the most efficient algorithms used nowadays in computational electromagnetics [1], [2]. Most of the developments in this area is concerned with the time domain techniques such as the finite difference time domain (FDTD) scheme. While it is quite straightforward to use Yee's mesh in the frequency domain [2], the finite difference frequency domain method (FDFD) enjoys smaller popularity, perhaps because it involves solving a large system of equations. There are, however, problems, such as filters, resonators and other high quality factor circuits, where the FDFD seems to be more appropriate than FDTD. Therefore, it seems worthwhile to develop methods which would make the FDFD technique computationally more efficient. One way toward this goal is to make the size of the system matrix as small as possible by applying the structure segmentation technique. Such an approach has been demonstrated, e.g., in [5] where the FDFD method has been combined with the mode matching technique via the generalized admittance matrix concept. This matrix has to be computed for the chosen subvolume by means of the FDFD method for each mode used in the mode matching analysis. In this contribution we propose a different approach in which the Helmholtz equation within entire structure is discretized by combining the partial eigenfunction expansion technique (PEE) [4] with the classical FDFD. As a result the matrix equation to be solved is very small and direct solution techniques can be applied.

The PEE (or modal expansion) has originally been proposed to accelerate the time domain analysis of waveguides [4] and waveguide discontinuities [3], [6]. In such problems one often deals with the situation where the computational domain contains long parts of a uniform waveguide, where transverse distribution of electromagnetic field can be evaluated analytically. One can take advantage of this fact and apply one dimensional discretization in those parts dividing uniform waveguide sections into slices and subsequently express the transverse field as a series of eigenfunctions or modes associated with cross-sectional geometry of the waveguide. The variables in PEE algorithm are the amplitudes of modes at a given slice rather than the values of electromagnetic field intensity at selected points in space. Combining the PEE analysis in homogeneous parts of the structure and FDTD in the rest of the circuit yields, as shown in [3], [4], [6], an improvement in numerical efficiency.

Due to the iterative character of FDTD algorithm, combining the eigenfunction expansion with the classical finite difference scheme in time domain is simple [3], [4]. Once the space decomposition has been carried out, different leap-frog algorithms are used in each subspace with fields at the boundaries computed at one time step with one algorithm serving as the boundary values for the next iteration for the other algorithm. The frequency domain formulation discussed in this letter is less straightforward as far as combining the FDFD part with the PEE part. This is because the matrix operator has to be set up explicitly.

I. Formulation of the Method

For concreteness, let us consider a capacitative iris in a waveguide shown in Fig. 1. The structure is divided into a region containing the iris, denoted by Ω_2, and parts of a homogeneous waveguide, denoted by Ω_1 and Ω_3. For region Ω_2 we define the classical FDFD operator [2], while the electromagnetic field in regions Ω_1 and Ω_3 is described according to the PEE scheme [4].

In the PEE algorithm the field in a uniform part of the structure is expressed as a superposition of modes. Hence,

$$\vec{E}(x,y,z) = \sum_{p=1}^{P} a_p(z)\vec{e}_p(x,y) \tag{1}$$

where $\vec{e}_p$ denotes the transverse distribution of the electric field of the pth mode (an eigenfuction) and $a_p(z)$ is the amplitude of this mode with respect to the direction of propagation z. Introducing the space discretization Δz in the z-direction and denoting the propagation constant of the free space as k_0, and the eigenvalue of the pth mode as k_p one gets the Helmholtz equation in the form

$$\frac{a_p^{n-1} - 2a_p^n + a_p^{n+1}}{\Delta z} - (k_0^2 - k_p^2)a_p^n = 0 \tag{2}$$

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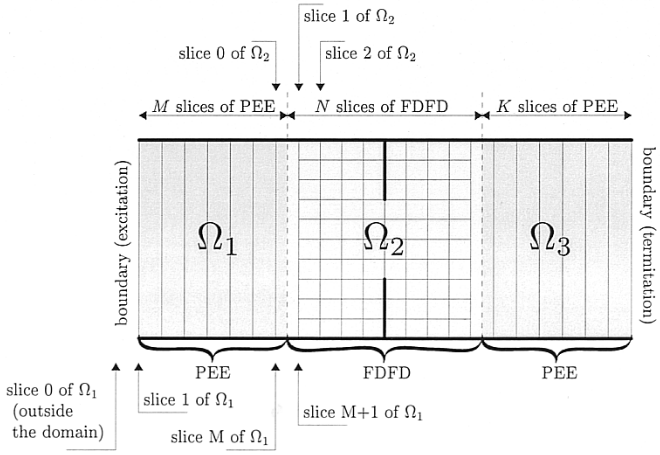

Fig. 1. Analyzed structure—a waveguide with an iris inside with division into slices and subdomains shown.

where a_p^n denotes the sample of amplitude (1) of pth mode at the nth slice of waveguide (Fig. 1). Introducing a vector $\boldsymbol{a}$ containing all a_p^n coefficients

$$\boldsymbol{a} = \left[a_1^1 \cdots a_P^1 \; a_1^2 \cdots a_P^2 \cdots a_1^M \cdots a_P^M\right] = \left[\boldsymbol{a}^1 \cdots \boldsymbol{a}^M\right] \quad (3)$$

the set of linear (2) for all P modes and N slices can be rewritten in a matrix form [2]

$$\boldsymbol{G}\boldsymbol{a} = 0. \quad (4)$$

To solve (4), one should give the boundary conditions: the a_p^0 amplitudes at the excitation plane and the a_p^{M+1} amplitudes at the end of Ω_1 for all p in (1). Vectors containing appropriate values can be defined as

$$\boldsymbol{a}^0 = \left[a_1^0 \cdots a_P^0\right] \quad \text{and} \quad (5)$$

$$\boldsymbol{a}^{M+1} = \left[a_1^{M+1} \cdots a_P^{M+1}\right]. \quad (6)$$

In Ω_2, the classical finite difference operator, denoted as $\boldsymbol{L}$, is defined [2]. The argument of $\boldsymbol{L}$, denoted as $\boldsymbol{v}$, can be written as

$$\boldsymbol{v} = \left[\boldsymbol{v}^1 \; \boldsymbol{v}^2 \cdots v^k \; \cdots \boldsymbol{v}^N\right] \quad \text{where} \quad (7)$$

$$\boldsymbol{v}^k = \left[e_{1,1,k}^x \; e_{1,1,k}^y \cdots e_{i,j,k}^y \cdots e_{I,J,k}^z\right] \quad (8)$$

assuming $I \times J$ nodes in the crossection and N in the direction of propagation. The x, y, z superscripts denote the component of the vector of electric field sampled in selected node of the mesh, $\boldsymbol{v}^i$ in (7) are vectors containing all field samples at the ith slice of Ω_2. The equation to be solved in region Ω_2 is

$$\boldsymbol{L}\boldsymbol{v} = 0 \quad (9)$$

subject to the boundary condition at the interface with Ω_1

$$\boldsymbol{v}^0 = \left[e_{1,1,0}^x \; e_{1,1,0}^y \; e_{1,1,0}^z \cdots e_{I,J,0}^z\right]. \quad (10)$$

The boundary condition $\boldsymbol{v}^{n+1}$ at the interface between Ω_2 and Ω_3 can be defined in similar way.

A. Assembling the Common Operator

The important fact is that the zero slice of Ω_2 is the same as M slice of Ω_1,and similarly, the first slice of Ω_2 is the boundary of Ω_1. To assemble the common the boundary conditions for $\boldsymbol{L}$ and $\boldsymbol{G}$ should be expressed in terms of the quantities of a neighboring region: vector $\boldsymbol{a}^{M+1}$ should be expressed by $\boldsymbol{v}^1$ and vector $\boldsymbol{v}^0$ by $\boldsymbol{a}^M$. According to (1) the following expression is satisfied:

$$\boldsymbol{v}^0 = \sum_{p=1}^{K} a_p^M \boldsymbol{e}_p \quad (11)$$

for the Ω_2 region (1), where $\boldsymbol{e}_p$ is a discretized pattern $\vec{e}_p(x, y)$ of the pth mode. Due to orthogonality of modes in the waveguide and the properties of inner product, the coefficients of $\boldsymbol{a}^{M+1}$ vector can be expressed as

$$a_p^{M+1} = \boldsymbol{e_p}^t \cdot \boldsymbol{v}^1 \quad (12)$$

where $^{\boldsymbol{t}}$ denotes the transposition and $\cdot$ is the standard inner product. Assembling (column-wise) $\boldsymbol{e}_p$ vectors multiplied by $k_0^2 - k_p^2$ into matrix $\boldsymbol{A}$ the common operator for all structure and its argument can be written as follows:

$$\left[\begin{array}{ccc|ccc} \ddots & & \vdots & & & \\ & \boldsymbol{G} & \vdots & & \boldsymbol{0} & \\ \cdots & \cdots & \ddots & \boldsymbol{A^t} & & \\ \hline & & \boldsymbol{A} & \ddots & \cdots & \cdots \\ & \boldsymbol{0} & & \vdots & \boldsymbol{L} & \\ & & & \vdots & & \ddots \end{array}\right] \cdot \left[\begin{array}{c} \vdots \\ \boldsymbol{a}^{M-2} \\ \boldsymbol{a}^{M-1} \\ \boldsymbol{a}^M \\ \hline \boldsymbol{v}^1 \\ \boldsymbol{v}^2 \\ \boldsymbol{v}^3 \\ \vdots \end{array}\right]. \quad (13)$$

B. Termination of the Computational Space

In order to terminate the computational space, the classical FD algorithm assumes a single mode propagation [7] so the analyzed structure has to be long enough to attenuate all higher order modes excited at the discontinuity. In the PEE-FDFD method every mode is terminated independently by simply putting

$$a_p^0 = a_p^1 \cdot \exp(-j\beta_p \Delta z) \qquad \beta_p = \sqrt{k_0^2 - k_p^2} \quad (14)$$

for all $i = (1, 2, \ldots P)$. A similar set of conditions can be written for the last slice in region Ω_3. The implementation of boundary condition resembles the modal techniques used in [9], [10]. However differencies between PEE-FDFD and the other techniques show where multiple doscontinuities are present. In this case the modal expansion is used at multiple slices.

II. Numerical Results

As a first example of the hybrid method PEE-FDFD we show the results of computations of the reflection coefficient for the fundamental mode TE_{10} in a waveguide containing an iris (Fig. 2). The structure is uniform in the y direction, so a 2-D formulation was used. The waveguide was discretized into

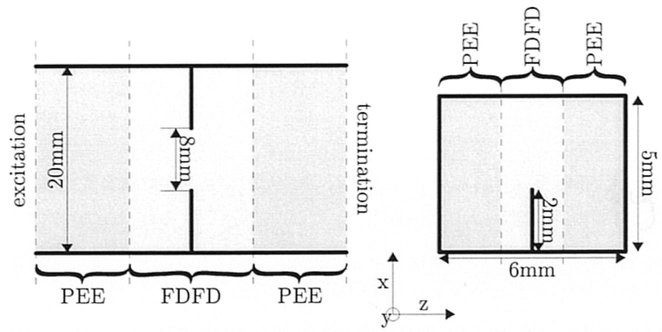

Fig. 2. Structures analyzed as examples, the waveguide with the iris (to the left) and the resonator with the wedge (to the right).

TABLE I
MAXIMUM ERROR OF COMPUTATION OF $|s_{11}|$ IN WAVEGUIDE WITH IRIS

	FDFD	Hybrid	Hybrid
acceleration	1	11	14
max. error [%]	0	1.72e-3	3.7e-3
no. of unknowns	1301	75	45
modes in PEE region	no PEE	15	19
structure length	71	5	3

20 cells in the x direction, the number of slices in z direction varied depending on the method. The dicretization step was $\Delta x = \Delta z = 1$ mm and the normalized frequency range f/f_0 was (1.05 – 1.95), where f_0 is the cutoff frequency of the TE_{10} mode. Note, that since there is only one discontinuity, all modes can be terminated close to the iris using (14), and hence the lengths of the PEE regions reduce to just one slice on each side. The length of the FDFD region varied. Table I compares of the number of unknowns, the solution time and the accuracy of the hybrid approach with respect the classical FDFD technique with a single mode termination. As one can see, for our problem we achieved a speedup of 14 with practically the same accuracy.

If the structure contains other discontinuities or reflecting planes then the length of the PEE region has to be larger than one slice. However, since the higher order modes are attenuated, the number of eigenfunctions in the PEE region can be made small. To illustrate this we have computed the resonant frequency of the first mode in a resonator structure shown in Fig. 2. For this mode the metal septum introduces field singularities and this causes significant errors in the finite difference analysis [8]. To achieve good accuracy the structure was finely meshed into 96 slices in the z direction and 80 cells in the x direction ($\Delta x = \Delta z = 1/16$ mm). The length of the FDFD region and the number of eigenfunctions in the PEE region were varied. As the distance from the FDFD region increased the number of modes was being gradually reduced to reach the minimum of just three modes at the slices next to the side walls. Table II shows the numerical data. Again, the errors and the solution time are referred to the FDFD calculations ($f_{\text{res}} = 19.57$ GHz). It is seen that very good results are obtained with the hybrid method much faster than with the classical FDFD method.

TABLE II
RELATIVE ERROR FOR THE FIRST MODE IN A RESONATOR WITH IRIS

	FDFD	Hybrid	Hybrid	Hybrid
acceleration	1	3.9	6.4	8.5
rel. error [%]	0	0.06	0.23	0.52
no. of unknowns	22625	4601	2769	2008
max. no. of modes in PEE	No PEE	10	15	20
No. of slices in FDFD region	96	15	7	3
No. of slices in PEE region	0	81	89	93

III. CONCLUSIONS

A frequency version of a hybrid method combining the PEE with the classical FDFD was introduced. Unlike other hybrid techniques that use modal expansion [9], [10], the proposed algorithm can be used both for termination of computational domain, and to accelerate computations in long parts of uniform guides between discontinuities.

REFERENCES

[1] K. S. Kunz and R. J. Lubbers, *Finite Difference Time Domain Method for Electromagnetics*. Boca Raton, FL: CRC, 1993.
[2] T. Weiland *et al.*, "Maxwell's grid equations," *Frequenz*, vol. 44, pp. 9–15, Jan. 1990.
[3] F. Alimenti, P. Mezzanotte, L. Roselli, and R. Sorrentino, "Efficient analysis of waveguide components by FDTD combined with time domain modal expansion," *IEEE Microwave Guided Wave Lett.*, vol. 5, pp. 351–353, Oct. 1995.
[4] M. Mrozowski, "A hybrid PEE-FDTD algorithm for accelerated time domain analysis of electromagnetic waves in shielded structures," *IEEE Microwave Guided Wave Lett.*, vol. 4, pp. 323–325, Oct. 1994.
[5] M. Mongiardo and R. Sorrentino, "Efficient and versatile analysis of structures by combined mode matching and finite difference methods," *IEEE Microwave Guided Wave Lett.*, vol. 3, pp. 241–243, Aug. 1993.
[6] A. Kreczkowski, T. Rutkowski, and M. Mrozowski, "Fast modal ABC's in the hybrid PEE-FDTD analysis of waveguide discontinuities," *IEEE Microwave Guided Wave Lett.*, vol. 9, pp. 186–188, May 1999.
[7] A. Christ and H. L. Hartnagel, "Three-dimensional finite-difference method for the analysis of microwave-device embedding," *IEEE Trans. Microwave Theory Techniques*, vol. 35, pp. 688–696, Aug. 1987.
[8] P. Przybyszewski and M. Mrozowski, "A conductive wedge in Yee's mesh," *IEEE Microwave Guided Wave Lett.*, vol. 8, pp. 66–68, Feb. 1998.
[9] G. Pelosi, R. Coccioli, and S. Selleri, *Quick Finite Element Method for Electromagnetic Waves*. Boston, MA: Artech House, 1998.
[10] G. Conciauro, M. Guglielmi, and G. Sorrentino, *Advanced Modal Analysis*. Chichester, U.K.: Wiley, 1999.